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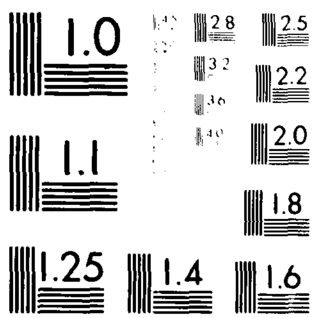

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WIENER-HOPF APPROACHES TO REGULATOR, FILTER/OBSERVER, AND OPTIMAL COUPLER PROBLEMS

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Most approaches to optimal linear stochastic control problems depend on time domain techniques for both their theoretical foundations and for computational algorithms. In contrast, this paper discusses innovations based on classical transform domain Wiener-Hopf theory. These innovations avoid several difficulties occurring in time domain solutions (e.g., those which arise in connection with singular regulator and filter problems). Emphasis will be upon solution techniques and properties in distinction to derivations.

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Abstract (Continued)

The following points will be demonstrated:

- Linear regulator, filter/observer problems can be solved using linear algebraic equations (in distinction to solving nonlinear Riccati equations).

- The R weighting matrix need not be positive definite nor is it necessary that R^{-1} exists. 1/R

- Singular regulator and filter/observer problems (e.g., the cheap control problem) can be handled neatly.

- The weighting matrices, R and Q, can be explicit functions of frequency. and

- There are some advantages in using non-diagonal Q and R matrices.

In addition, a new class of problem is solved using the Wiener-Hopf approach. The problem treats the case of a continuous performance index, together with a continuous plant and sampled measurements to specify both the optimal (discrete) digital control law and the optimal (continuous) data hold to be used for coupling the digital control law to the plant actuators.

FOREWORD

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The author wishes to express his deep appreciation to L. G. Hofmann for the many stimulating discussions which lead to a clearer understanding of the manner in which time and frequency domain approaches cope with singular problems.

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SECTION I

INTRODUCTION

The objective of the research effort reported herein was to develop, via the Wiener-Hopf (W-H) approach, an optimal closed-loop solution for digital control of continuous plants using a continuous cost function. The importance of the problem class resides in two facts:

- Solution consists of two parts
  - optimal discrete control law
  - optimal continuous data hold
- Use of a continuous cost function assures accountability for the inter-sample behavior of the continuous plant response.

This is worthy of further elaboration. The use of a continuous cost function in conjunction with a continuous plant model yields a control law that is optimal at all instants in time. However, the only constraint placed on the closed-loop system response is at the sampling instances when a discrete cost function is employed. The "inter-sample" behavior of the continuous plant response can be very unsatisfactory. This is especially so if the data rate is low and the open-loop plant contains lightly damped modes. Also, when a discrete cost function is minimized, the data hold (the coupler between the computer and the control actuators) is specified arbitrarily by the designer (more often than not, a zero-order data hold is utilized). On the other hand, if a continuous cost function is minimized via the Wiener-Hopf approach, the optimal solution specifies both the control law and the optimal form of the coupler.

As will be shown in Section IV, the combined coupler-control law problem yields a Wiener-Hopf equation that is more formidable than its discrete or continuous counterparts. Therefore, it is worthwhile to comment on the issues involved in pursuing a solution using this approach.

The advantages of using modern matrix Wiener-Hopf minimization procedures are not widely appreciated in today's control community. The primary reason for this is that many experienced scientists and engineers have been alienated by the difficulty in applying the spectral factorization solution method which Wiener used (Ref. 1). Very few optimal control practitioners are cognizant of the fact that Wiener-Hopf equations can be solved by a direct solution technique which makes spectral factorization unnecessary (Refs. 2 and 3) and uses mathematics no more difficult than partial fraction expansions. Moreover, the method handles the multi-controller cases as well as unstable, nonminimum phase plants. We believe that increased awareness of the basic and simple methods available for solving Wiener-Hopf equations can increase modern control engineering productivity and will encourage a more integrated use of time domain and frequency domain techniques for problem solution.

The technical development proceeds in the sections which follow. Section II reviews the W-H method for solving the continuous regulator problem. This is not strictly necessary since the basic solution principles for the regulator are contained in Ref. 3. However, a clear understanding of the manner in which the direct method works will make the solution method used in the optimal coupler problem easier to follow. Moreover, a review of the regulator case affords the opportunity to make clear that the W-H approach provides a unifying framework wherein no modifications to the basic approach are required in order to treat singular cases. In particular, the "bulleted" items in the abstract will be clarified using a series of numerically tractable examples.

Section III deals with the optimal linear continuous stochastic control problem via the Wiener-Hopf formulation. It is necessary to treat this case because there is no established literature on solution of these optimization problems by means of Wiener-Hopf techniques. First, the time domain form of the LQG optimal stochastic control problem is treated using frequency domain methods. It is shown that the steady-state gains of the Kalman filter can be found using the W-H approach and linear solution methods. That is, it is not necessary to solve a nonlinear Riccati equation. Next, an alternative W-H formulation is postulated which does not make use of the separability principle. This leads to a rather interesting formulation in that two spectral matrices are involved. One is recognizable as being the regulator spectral matrix, while the other is recognized as the spectral matrix associated with the filter/observer problem. It is shown that this W-H formulation survives, without singularity, the limiting condition wherein the measurement noise vector is set identically equal to zero. This yields the optimal observer solution without the necessity of resorting to limiting forms or special partitioning (for example, Refs. 4 and 5).

Section IV treats the optimal coupler problem using an extension of the second approach discussed in Section III (i.e., the formulation which does not make use of the separability principle). Again, there will be two spectral matrices which appear in the W-H equation; the "regulator" spectral matrix remains a function of the complex frequency variable s, but now the "filter observer" spectral matrix will depend on the delay operator $z = e^{sT}$.

It must be noted that the situation with respect to the design of a digital controller using a continuous cost function will suffer the same shortcomings presently associated with continuous regulator design. That is, questions pertaining to the selection of the Q's and R's of the regulator weighting matrices (so as to produce designs which are not only "optimal" but "satisfactory" as well) will obviously persist for the optimal coupler problem. In this regard, it is hoped that the examples presented herein (which emphasize the use of non-diagonal Q and R, $R < 0$, etc.) will encourage optimal control practitioners to ease self-imposed restrictions placed on the weighting matrices.

SECTION II

LINEAR REGULATOR PROBLEM

The open-loop plant model used throughout the paper will be

$$\begin{aligned} \dot{x} &= Fx + Gu \\ X(s) &= [Is - F]^{-1} GU + [Is - F]^{-1} x(o) \qquad (1) \\ &= AU + Bx(o) \end{aligned}$$

Assuming a control law of the form

$$U = -KX \qquad (2)$$

results in the closed-loop configuration shown in Fig. 1. (K is not restricted to being only a gain matrix.) The continuous cost function:

$$J = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} (X_* QX + U_* RU)\, ds \qquad (3)$$

$$X_* = X'(-s) \quad ; \quad U_* = U'(-s)$$

is minimized by taking a variation on U such that

$$U = U_0 + \lambda U_1 \qquad (4)$$

where U_0 is the optimal control and U_1 is any physically realizable (exists for $t \geq 0$) but arbitrary variation. The first variation gives a <u>necessary</u> condition for an optimum (refer to Ref. 3 for the details) as:

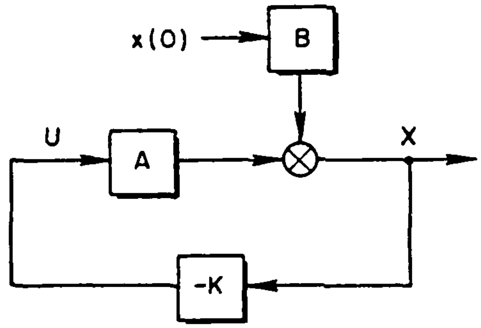

Figure 1. Closed-Loop Regulator

$$J_c = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} U_{1*}\,\psi(s)\,ds$$

$$= \int_{-\infty}^{\infty} u_1'(t)\,\psi(t)\,dt = 0 \tag{5}$$

where

$$[R + A_*\mathcal{Q}A]U_0 + A_*\mathcal{Q}B\,x(0) = \psi \tag{6}$$

Since $u_1'(t)$ exists for $t \geq 0$, a <u>sufficient</u> condition for satisfying Eq. 5 is that $\psi(t)$ exists only for $t < 0$; hence, the product $u_1'(t)\,\psi(t)$ is identically zero.

The second variation,

$$J_d = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} U_{1*}[R + A_*\mathcal{Q}A]U_1\,ds \tag{7}$$

can then be investigated to see if the extremum obtained is truly a minimum ($J_d > 0$) or a maximum ($J_d < 0$). Note that c in Eq. 7 is not necessarily the same as the c in Eq. 5. Since $u'(t)$ exists for positive time, its laplace (two-sided) transform exists in the s-plane in a domain where $\sigma > \sigma_1$. On the other hand, $\psi(t)$ exists for $t \leq 0$, and its transform exists in the s-plane in a domain where $\sigma < \sigma_2$. Therefore, c (which defines the poles enclosed by the contour) must satisfy the condition

$$c < \min \sigma_2, \sigma_1 \tag{8}$$

in order that U_{1*} and ψ have a common strip of convergence. For Eq. 5 to be identically zero, the frequency domain requirement is to close a contour to the left and enclose no poles; therefore, the sum of the residues (the value of the integral) is zero. The path traversed parallel to the $j\omega$-axis is determined by the value of c in Eq. 8 and is not necessarily the $j\omega$-axis itself ($c = 0$).

To insure $\psi(s)$ exists for $\sigma < \sigma_2$ (e.g., exists in some left half plane), we must pick U_0 in such a manner that any poles of Eq. 6 which exist in some right half plane (e.g., positive time functions) cancel identically into each and every numerator of Eq. 6.

U_0 will have the form

$$U_0 = W(s)x(0) \tag{9}$$

The required compensation can be computed as

$$K = -W[B + AW]^{-1} \tag{10}$$

For the special case where K is a gain matrix, it suffices to use the initial value theorem:

$$K = \lim_{s \to \infty} -sW(s) \tag{11}$$

Finally, it is not necessary to compute the closed-loop transfer functions using

$$X(s) = [Is - F + GK]^{-1}x(0) \tag{12}$$

since

$$X = [B + AW]\ x(0) \tag{13}$$

will work just as well. In fact, Eq. 13 persists even in the singular cases where the K matrix has infinite entries.

What are the candidate poles for U_O? These are picked from (Refs. 2, 3):

$$\det[R + A_*QA] \equiv (D\bar{D})^{P-1}\ \Delta\bar{\Delta} \tag{14}$$

where P is the number of controllers, D represents the open-loop poles, and Δ represents the closed-loop poles.

What are the candidate zeros of U_O? These are unknown. Therefore, one simply specifies polynomials with <u>unknown</u> <u>coefficients</u>. The number of unknown coefficients is equal to the number of "positive time" poles which must be cancelled. More precise details are available in Refs. 2 and 3; our thrust here is to demonstrate how the simple principle of positive pole cancellation yields the solution without the need to factorize $R + A_*QA$.

A. SINGLE CONTROL POINT EXAMPLE

Let

$$\dot{x} = 2x + 3u$$

so that

$$X(s) = \frac{3U(s)}{s-2} + \frac{1}{s-2}\,x(0) = AU + Bx(0)$$

Notice that the unstable open-loop pole exists for $\sigma > 2$. Suppose $R = 1$, $Q = -1/3$ $(Q < 0)$, then the W-H equation

$$[R + A_* QA]U_0 + A_* QBx(0)$$

$$= \left[1 + \frac{3(-1/3)}{(-s-2)} \frac{3}{(s-2)} \right] U_0$$

$$+ \frac{3(-1/3)}{(-s-2)} \frac{(1)}{(s-2)} x_0$$

$$= \psi \tag{15}$$

or

$$\left(\frac{-s^2+1}{-s^2+4}\right)U_0 - \frac{1}{-s^2+4} x(0) = \frac{\Delta\bar{\Delta}}{D\bar{D}}U_0 - \frac{x(0)}{D\bar{D}} \tag{15}$$

Since $\det[R + A_*QA] = (-s+1)(s+1)$, the optimal closed-loop pole is at $s = -1$. Let $U_0 = \xi/\Delta = \xi/(s+1)$ and substitute into Eq. 15:

$$\frac{(-s+1)(s+1)}{-s^2+4} \frac{\xi}{s+1} - \frac{1}{-s^2+4} x(0) = \psi \tag{16}$$

Only one positive time function pole $(s-2)$ survives in Eq. 16; therefore, only one unknown coefficient is needed:

$$\frac{(-s+1)a_0 - x(0)}{(-s^2+4)} = \psi$$

The numerator must contain $(s-2)$; therefore, the numerator equals zero when $s = 2$.

$$(-s+1)a_0\Big|_{s=2} = x(0) \Rightarrow a_0 = -x(0)$$

Thus

$$U_0 = W\,x(0) = \frac{-1}{s+1}\,x(0) \tag{17a}$$

$$\psi = \frac{x(0)}{-s-2} \quad , \quad \sigma < 2 \tag{17b}$$

$$K = \lim_{s \to \infty} -sW(s) = 1 \tag{17c}$$

$$X = [B + AW]\,x(0)$$

$$= \left[\frac{1}{s-2} + \frac{3}{s-2}\left(\frac{-1}{s+1}\right)\right] x(0) = \frac{1}{s+1}\,x(0) \tag{17d}$$

Evaluating the first variation

$$J_c = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} U_{1*}\,\psi(s)\,ds$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} U_{1*}\,\frac{1}{(-s-2)}\,ds$$

$$\equiv 0 \qquad\qquad c < \min -2, -\sigma_1$$

To see that $J_c \equiv 0$, consider any realizable U_1 (not necessarily stable) that exists for $\sigma > \sigma_1$. The U_{1*} exists for $\sigma < -\sigma_1$ and the c selected is the minimum of -2 or $-\sigma_1$. This is sketched in Fig. 2. Closing the contour to the left using $c < \min \sigma_1, -2$ encloses no poles. The summation of the residue is zero, and the value of the first variation is identically zero. Consider next the second variation:

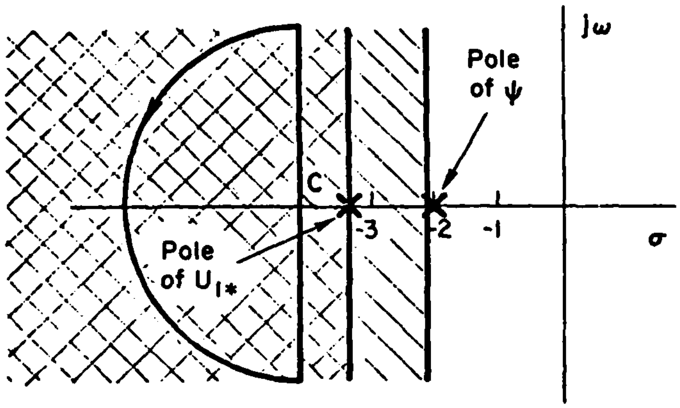

Figure 2. s-Plane Domains

$$J_d = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} U_{1*}[R + A_* Q A]U_1 \; ds$$

$$= \frac{1}{2+j}\int_{c-j\infty}^{c+j\infty} \left(U_{1*}\,\frac{\bar{\Delta}}{\bar{D}}\,\frac{\Delta}{D}\,U_1\right) ds$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F_* F \; ds$$

$$= \int_0^\infty f^2 \, dt > 0 \tag{18}$$

An interesting observation now can be made regarding the domains of existence when the open-loop plant is unstable. $D = (s - 2)$ requires $\sigma < -2$, and therefore there is no common strip of convergence. At this point two options are available. The admissible U_1 can be restricted so that $\Delta U_1/D$ has a common strip of convergence with $U_{1*}\bar{\Delta}/\bar{D}$, insuring that Eq. 18 is both positive and finite. On the other hand, the only real issue is whether or not the second variation is positive. Thus, there is no need to restrict U_1 if one is willing to accept a positive, if unbounded, second variation.

B. SIMPLIFIED W-H CONDITIONS

It is not necessary to work with the complete W-H description as given in Eq. 6. Equation 6 can be reduced to two computationally simpler requirements.

Since

$$\det[R + A_*QA] \equiv (D\bar{D})^{P-1}\,\Delta\bar{\Delta} \tag{19}$$

let

$$U_0 = \xi(s)/\Delta \tag{20}$$

where $\xi(s)$ is an unknown polynomial matrix. Further, describe the plant matrices, A and B, in terms of the open-loop poles and their adjoint matrices. That is,

$$A = \frac{A^a}{D}\quad,\quad B = \frac{B^a}{D}\quad,\quad D = \text{Open-loop poles} \tag{21}$$

The W-H equation becomes:

$$\frac{[RD\bar{D} + A_*^aQA^a]\xi(s) + [\Delta A_*^aQB^a]x(0)}{D\bar{D}\Delta} = \psi \tag{22}$$

From the previous discussion it is clear that each and every numerator of ψ must contain D and Δ. Therefore, each numerator of ψ must be zero for those values of s such that $D = \Delta = 0$. (The poles of D may be located <u>anywhere</u> in the s plane. It is the fact that they exist in some half plane for which $\sigma > \sigma_1$ which "tags" the roots of D as giving rise to positive time functions.) When $D = 0$, Eq. 22 reduces to

$$A_*^aQ[A^a\xi(s) + \Delta B^a x(0)] = 0 \tag{23}$$

Since $A_*^a Q$ is already analytic in some left half plane, it need not be considered. Thus, we have a first W-H condition.

1st W-H Condition:

$$A^a \xi(s) + \Delta B^a x(0) = 0 \quad , \quad D = 0 \tag{24}$$

Next, let $\Delta = 0$ in Eq. 22 and obtain the second W-H condition:

2nd W-H condition:

$$[RD\bar{D} + A_*^a Q A^a]\xi(s) = 0 \quad , \quad \Delta = 0 \tag{25}$$

Notice that the first W-H condition persists even in the limiting case of $Q \rightarrow 0$. Furthermore, even though Eqs. 24 and 25 are <u>matrix</u> equations, it usually suffices to pick only <u>one</u> component of each since Eq. 19 assures linear dependence when $D = \Delta = 0$. For example, using the first component of Eq. 24 will produce the same linear set of equations, in terms of the unknown entires of $\xi(s)$, as will the second (or third) component of Eq. 24. Finally, note that nowhere does R^{-1} appear in the W-H conditions — a fact that makes the direct solution method ideal for the evaluation of singular cases. A three-state, two-control-point example will be used to clarify the mathematical details.

Let

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ -4 & 2 \\ 1 & 0 \end{bmatrix} u \tag{26a}$$

$$X = AU + Bx(0) = \frac{\begin{bmatrix} -4s+1 & (s+1)^2 \\ s(-4s+1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)} U(s)$$

$$+ \frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)} x(0) \quad (26b)$$

Suppose

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 30 \end{bmatrix}, \quad Q = \begin{bmatrix} 94 & & \\ & 0 & \\ & & -705 \end{bmatrix} \quad (27)$$

so that $R < 0$, Q^{-1} does not exist. This is a so-called "cheap control problem," since there is no weight on the first controller. A routine but tedious computation gives:

$$R + A_*QA = -\frac{\left[\begin{array}{c|c} 705s^4 + 2209s^2 + 611 & 329s^3 + 564s^2 + 846s + 611 \\ \hline -329s^3 + 564s^2 - 846s + 611 & 30s^6 + 146s^4 - 37s^2 + 611 \end{array}\right]}{D\bar{D}}, \quad D = s(s^2+4) \quad (28)$$

Evaluating the determinant of $R + A_*QA$ we find

$$\det[R + A_*QA] \propto s^2 + 2s + 2 = (s+1)^2 + (1)^2 = \Delta \, * \quad (29)$$

---

*Read the symbol $\propto$ as "contains the factor".

Since there is no weight on U_1, we find that the order of the closed-loop system has been reduced by one. In order to force both D and Δ to cancel into ψ now requires only 5 unknown coefficients rather than 6 (D is third order, Δ is second order). Let

$$U_0 = \frac{\begin{bmatrix} a_0 s^2 + a_1 s + a_2 \\ b_1 s + b_2 \end{bmatrix}}{s^2 + 2s + 2} x(0) = W x(0) \qquad (30)$$

The "burden" of the extra coefficient is assumed by the first controller, which has been excluded from the performance index. The feedback gains from X to U will be infinite, since U is not proper-rational. Consider the W-H condition; only one component of each matrix need be considered. Evaluate the first W-H condition for D = 0 and obtain three equations in the five unknowns. Then evaluate the second W-H condition to get two equations in the five unknowns (for complex roots equate real to real and imaginary to imaginary).

1st W-H Condition:

$$\begin{aligned} &(-4s + 1)(a_0 s^2 + a_1 s + a_2) + (s^2 + 2s + 1)(b_1 s + b_2) \\ &\quad = -\Delta[\, s^2 - 1 \;\vdots\; s + 1 \;\vdots\; 5\,]\, x(0) \qquad (31) \\ &\qquad s = 0, +2j \qquad (D = 0) \end{aligned}$$

2nd W-H Condition:

$$\begin{aligned} &(705s^4 + 2209s^2 + 611)(a_0 s^2 + a_1 s + a_2) \\ &\quad + (329s^3 + 564s^2 + 846s + 611)(b_1 s + b_2) = 0 \qquad (32) \\ &\qquad s = -1 + j \qquad (\Delta = 0) \end{aligned}$$

We obtain five equations in five unknowns:

$$
\begin{array}{l}
s = 2j,\ \text{Real} \\
s = 2j,\ \text{Imag} \\
s = 0 \\
s = -1 + j,\ \text{Real} \\
s = -1 + j,\ \text{Imaj}
\end{array}
\begin{bmatrix}
-4 & 16 & 1 & -8 & -3 \\
32 & 2 & -8 & -6 & 4 \\
0 & 0 & 1 & 0 & 1 \\
-8836 & 6627 & -2209 & -799 & 423 \\
4418 & 2209 & -4418 & 47 & 376
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}
$$

$$
= \begin{bmatrix}
-10 & 10 & 10 \\
20 & 0 & -20 \\
2 & -2 & -10 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} x(0) \qquad (33a)
$$

The solution is:

$$
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \frac{\begin{bmatrix}
-136 & -18 & -327 \\
-425 & -60 & -750 \\
-289 & -87 & -858 \\
-799 & -282 & -1128 \\
799 & -423 & -1692
\end{bmatrix}}{255}
\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \qquad (33b)
$$

Placing Eq. 33b in the U = Wx(0) format gives:

$$U = W(s)\,x(0)$$

$$= \frac{1}{255}\,\frac{\left[\begin{array}{c|c|c} -136s^2-425s & -18s^2-60s & -327s^2-750s \\ -289 & -87 & -858 \\ \hline -799s+799 & -282s-423 & -1128s-1692 \end{array}\right]}{(s^2+2s+2)}\,x(0) \tag{34}$$

There will be three infinite feedback gains (refer to Eq. 11) since the first controller is not proper rational.

$$K = \frac{1}{255}\begin{bmatrix} 136k_o & 18k_o & 327k_o \\ 799 & 282 & 1128 \end{bmatrix}_{\lim\, k_o \to \infty} \tag{35}$$

Note that the first controller gains go to infinity, at a rate determined by the ratios of the entries of the first row. Even though the gains are infinite, the closed-loop transfer functions are readily computed:

$$X = [B + AW]\,x(0)$$

$$= \frac{\begin{bmatrix} 255(s+1) & 45 & 180 \\ 544s-1309 & 327s+423 & 1308s+1692 \\ -136s-34 & -18s-42 & -72s-168 \end{bmatrix}}{255(s^2+2s+2)}\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \tag{36}$$

The first W-H condition guarantees that D, the open-loop poles, cancel into each numerator of [B + AW].

This example has been carried through in careful detail and demonstrates all facets of the direct approach. Other examples, which demonstrate the "bulleted" items in the abstract, are synopsized in the appendix. These examples further demonstrate that the direct solution method continues to work, without need for <u>modification</u>, regardless of the form of R and Q.

SECTION III

WIENER-HOPF FORMULATION — OPTIMAL LINEAR STOCHASTIC CONTROL

The open-loop plant equation is modified by the addition of a process noise vector n and an output equation:

$$\begin{aligned} \dot{x} &= Fx + Gu + n \quad , \quad x(0) = x_0 \\ y &= Hx + v \end{aligned} \tag{37}$$

The transform of Eq. 37 is:

$$\begin{aligned} X(s) &= [Is - F]^{-1}GU(s) + [Is - F]^{-1}[N + x_0] \\ &= A(s)U(s) + B(s)[N(s) + x_0] \end{aligned} \tag{38}$$

The block diagram of the open-loop plant is shown in Fig. 3.

The time domain formulation of the linear optimal stochastic control problem is given in Fig. 4 (Ref. 6) and the equivalent frequency domain formulation is given in Fig. 5.

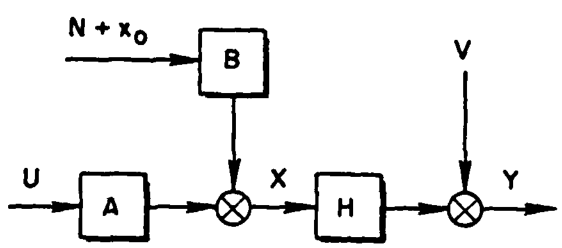

Figure 3. Block Diagram for the Plant and Measurements

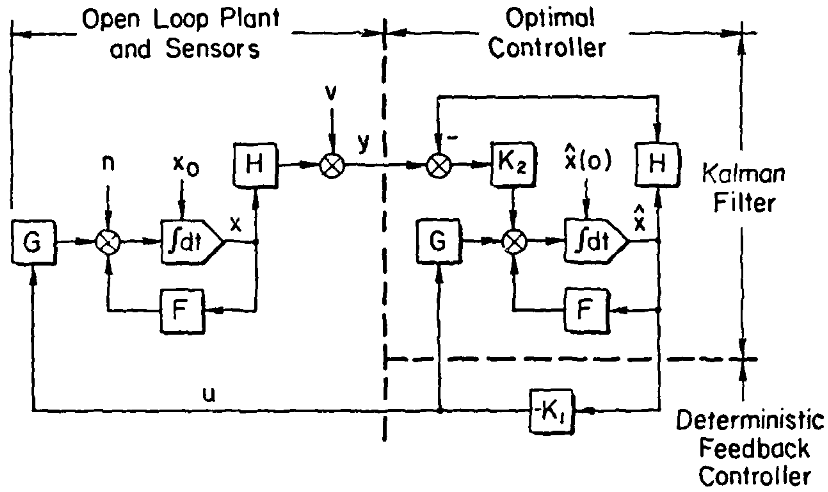

Figure 4. Time Domain Formulation of the Linear Optimal Stochastic Control Problem

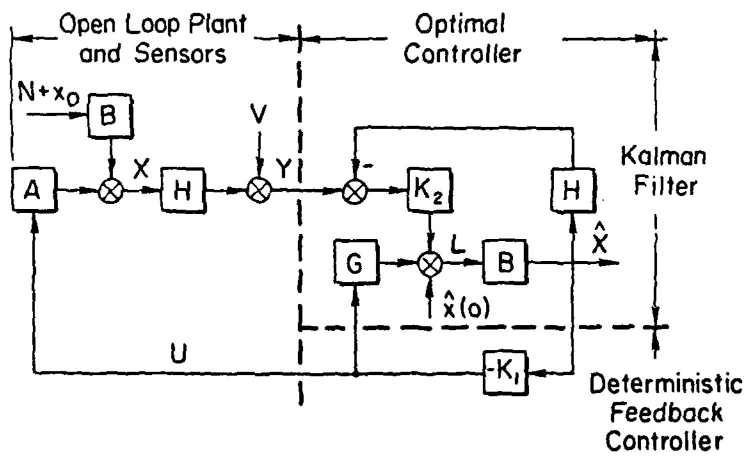

Figure 5. Frequency Domain Formulation of the Linear Optimal Stochastic Control Problem

A separability principle formulation is first used. That is, the K_1 matrix represents the regulator gains, and now the task is to find the Kalman filter gains K_2. This will be done by minimizing the mean square error between the state vector X and its estimate, $\hat{x}$. Using Fig. 5, solve for L and write $\hat{X} = BL$:

$$\hat{X} = B[I + K_2HB]^{-1}K_2Y + B[I + K_2HB]^{-1}GU \tag{39}$$

Let

$$W_a = B[I + K_2HB]^{-1}K_2 \tag{40}$$

and write the error as

$$E = \hat{X} - X = W_a[V + HB(N + x_o)] - B[N + x_o]^{*} \tag{41}$$

Forming E_*E and taking the gradient with respect to W_{a_*} gives the W-H equation (see Ref. 5 or refer to Appendix C):

$$W_a[q_{\overline{VV}'} + HBq_{\overline{NN}'}B_*H_*] - Bq_{\overline{NN}'}B_*H_* = \psi \tag{42}$$

where $q_{\overline{VV}'}$ is the auto spectra of the measurement noise and $q_{\overline{NN}'}$ is the auto spectra of the process noise. The assumption of uncorrelated N and V was used in arriving at Eq. 42, although the more general case is easily treated. Furthermore, $x(0)$ and $\hat{x}(0)$ have been set equal to zero in order to simplify the presentation. To compensate for this shortcoming, solution of the output regulator problem, via W-H, is given in Appendix D.

$$
\begin{aligned}
{}^{*}\quad \hat{X} - X &= W_aY + B[I + K_2HB]^{-1}GU - AU - B[N + x_o] \quad ; \quad A = BG \\
&= W_aY + B[I + K_2HB]^{-1}[I - I - K_2HB]GU - B[N + x_o] \\
&= W_aY - W_aHAU - B[N + x_o] \\
&= W_a[V + HB(N + x_o) + HAU - HAU] - B[N + x_o]
\end{aligned}
$$

Therefore,

$$\hat{X} - X = W_a[V + HB(N + x_o)] - B[N + x_o]$$

Observe that now the unknown of the W-H equation, W_a, is postmultiplied by a spectral matrix which we will term the filter/observer spectral matrix. Equation 42 can be solved, using the direct solution method, in exactly the same manner as the regulator problem of the previous section. Moreover, once W_a is found, the K_2 feedforward matrix can be computed using

$$K_2 = B^{-1}W_a[I - HW_a]^{-1}$$

A simpler result is obtainable, if one considers the properties of W_a as a solution to a W-H equation. Rewrite Eq. 43 as

$$K_2[I - HW_a] = [Is - F]W_a \tag{44}$$

and observe that, when W_a is a proper rational function, one can find K_2 directly by letting $s \rightarrow \infty$.

$$K_2 = \lim_{s \rightarrow \infty} sW_a(s) \tag{45}$$

One cannot use Eq. 45 in the singular cases which occur when some or all of the measurement noise components are zero. In this event, $(I - HW_a)$ is singular.

Once either K_2 or W_a is known, the controller to measurable output transfer functions can be computed using (refer to Fig. 5):

$$U = -K_1[I + (I - W_aH)AK_1]^{-1}W_aY \quad * \tag{46}$$

---

* $\hat{X} = W_aY - B[I + K_2HB]^{-1}GK_1\,\hat{X}$

or

$$\begin{aligned}
\hat{X} &= [I + B\,(I + K_2HB)^{-1}GK_1]^{-1}\,W_aY \\
&= [I + AK_1 - BGK_1 + B(I + K_2HB)^{-1}\,GK_1]^{-1}\,W_aY \\
&= [I + AK_1 - B(I + K_2HB)^{-1}(I + K_2HB - I)GK_1]^{-1}W_aY \\
&= [I + AK_1 - W_aHAK_1]W_aY
\end{aligned}$$

or

$$U = -K_1[Is - F + GK_1 + K_2H]^{-1} K_2Y \tag{47}$$

Clearly, Eq. 46 persists even when the feedforward gain matrix K_2 is singular.

Equation 46 gives an interesting limiting form. Suppose $\varphi_{\overline{VV}'} \equiv 0$ (no measurement noise) and H is invertable. Then $W_a = H^{-1}$ satisfies Eq. 42. But

$$W_a = H^{-1} \Longrightarrow U = -K_1H^{-1}Y \quad . \tag{48}$$

The following illustrative example clarifies the mathematical details.

A. ILLUSTRATIVE EXAMPLE, $\varphi_{\overline{VV}'} \neq 0$

Suppose the open-loop plant is described by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ -4 & 2 \\ 1 & 0 \end{bmatrix} u + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{49}$$

and Y = HX + V, giving

$$X = AU + B[N + x(0)] = \frac{\begin{bmatrix} -4s+1 & s^2+2s+1 \\ s(-4s+1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$+ \frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)} [N + x(0)] \tag{50}$$

Let

$$R = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad Q = \begin{bmatrix} 2 & -1 & 5 \\ -1 & 0 & 2 \\ 5 & 2 & 18 \end{bmatrix} \tag{51}$$

First, solve for the regulator gains; call this matrix K_1. Using the direct methods of the previous section, the solution of the regulator W-H equation gives

$$U_o = Wx(0) = -\frac{\begin{bmatrix} s^2+4s+9 & 0 & 2s^2-2 \\ 2s^2+3s+9 & s^2+5s+6 & 6s^2+7s+32 \end{bmatrix}}{(s+1)(s+2)(s+3)} x(0) \tag{52}$$

Application of the initial value theorem gives the feedback gains as

$$K_1 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 6 \end{bmatrix} \tag{53}$$

Further, the matrix of "regulator" closed-loop transfer functions is computed as

$$X = [B + AW]\, x(0)$$

$$= \frac{\begin{bmatrix} (s+1)(s+3) & 0 & -6(s+1) \\ -5(s+3) & (s+2)(s+3) & s+32 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}}{(s+1)(s+2)(s+3)} x(0) \tag{54}$$

To differentiate between the regulator poles and filter/observer poles, let the regulator poles be

$$\Delta_1 = (s+1)(s+2)(s+3) \tag{55}$$

The second step is to find the Wiener filter W_a, which is tantamount to specifying the Kalman gains, K_2. The filter portion of the problem has the W-H equation

$$W_a[\varphi_{\overline{V}V'} + HB\varphi_{\overline{N}N'} B_* H_*] - B\varphi_{\overline{N}N'} B_* H_* = \psi \tag{56}$$

For demonstration purposes, assume only one output measurement. Let

$$Y = HX + V \quad , \quad H = [1, \; 0, \; 0] \tag{57}$$

and also let the noise sources be described by the intensities

$$\varphi_{\overline{V}V'} = 1 \quad , \quad \varphi_{\overline{N}N'} = \begin{bmatrix} 85 & 0 & 0 \\ 0 & 1690 & 0 \\ 0 & 0 & 505 \end{bmatrix} \tag{58}$$

A straightforward computation gives

$$\varphi_{\overline{V}V'} + H B \varphi_{\overline{N}N'} B_* H_* = \frac{\Delta_2 \overline{\Delta}_2}{D\overline{D}} \tag{59}$$

where (specifying Δ_2 as the filter poles)

$$\begin{aligned} \Delta_2 &= (s + 4)(s + 5)(s + 6) \\ \overline{\Delta}_2 &= -s^3 + 15s^2 - 74s + 120 \end{aligned} \tag{60}$$

Also, one finds

$$B\varphi_{\overline{N}N'} B_* H_* = \frac{\begin{bmatrix} 85s^4 - 1860s^2 + 14,400 \\ -2115s^3 + 14,740s \\ 85s^3 + 2440s^2 - 4300s + 14,400 \end{bmatrix}}{D\overline{D}} \tag{61}$$

The W-H equation has the form

$$\frac{\begin{bmatrix} a_0 s^2 + a_1 s + a_2 \\ b_0 s^2 + b_1 s + b_2 \\ c_0 s^2 + c_1 s + c_2 \end{bmatrix}}{\cancel{\Delta}_2} \; \frac{\cancel{\Delta}_2 \bar{\Delta}_2}{D\bar{D}}$$

$$- \frac{\begin{bmatrix} 85s^4 - 1860s^2 + 14,400 \\ -2115s^3 + 14,740s \\ 85s^3 + 2440s^2 - 4300s + 14,400 \end{bmatrix}}{D\bar{D}} = \psi \qquad (62)$$

Since each component of ψ must contain D, the open-loop roots, we require nine unknown coefficients to force the cancellation. Of course, each ψ component can be treated separately, so that the basic problem is solving three equations for three unknowns, rather than nine equations for nine unknowns. Letting s = 0,2j gives these equations; for example,

$$\left(a_0 s^2 + a_1 s + a_2\right)\Big|_{s=0,2j}$$

$$= \frac{85s^4 - 1860s^2 + 14,400}{-s^3 + 15s^2 - 74s + 120}\Bigg|_{s=0,2j}$$

The final result is

$$W_a = \frac{\begin{bmatrix} 15s^2 + 70s + 120 \\ 70s^2 + 60s \\ 13s^2 + 2s + 120 \end{bmatrix}}{(s+2)(s+4)(s+5)} \qquad (63)$$

Finding the Kalman gains is now a simple task, since

$$K_2 = \lim_{s \to \infty} sW_a(s) = \begin{bmatrix} 15 \\ 70 \\ 13 \end{bmatrix} \tag{64}$$

Once __either__ W_a or K_2 is known, the input/output transfer functions can be computed using Eqs. 46 or 47.

$$U = - \frac{\begin{bmatrix} 41s^2 + 60s + 19 \\ 178s^2 + 486s + 701 \end{bmatrix}}{s^3 + 21s^2 - 7s - 27} Y$$

$$= - \begin{bmatrix} \dfrac{41s + 19}{s^2 + 20s - 27} \\ \dfrac{178s^2 + 486s + 701}{(s + 1)(s^2 + 20s - 27)} \end{bmatrix} Y \tag{65}$$

It comes as no surprise that the transfer functions describing the input/output relationships between the controllers and measurable outputs may well be unstable, as they are in this example.

Singular cases can also be treated using the direct method since Eqs. 42 and 46 can always be solved.

B. SOLUTION WITHOUT THE SEPARABILITY PRINCIPLE

The linear stochastic optimal control problem can be solved without making use of the separability principle. This can be very useful when:

- The regulator part of the solution is singular ($K_1 \to \infty$).

- It is desirable to reduce the computational burden (e.g., if one has a very high order system but only a few measurable outputs and controllers, it is more reasonable to compute a few transfer functions than attempt to estimate a large number of states).

We may proceed according to Fig. 6.

In Fig. 6 we desire to minimize the performance index

$$J = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \widetilde{(X_* QX + U_* RU)}\, ds \tag{66}$$

where $\widetilde{(\cdot)}$ denotes the expected value of $(\cdot)$.

First, form the integrand of the performance index

$$\Phi = X_* QX + U_* RU \tag{67}$$

using the equations (refer to Fig. 6)

$$U = -[I + WHA]^{-1} W[V + HBN] \tag{68}$$

$$X = BN + AU = BN - A[I + WHA]^{-1} W[V + HBN] \tag{69}$$

Note, from Eq. 68, that the closed-loop stability is determined by

$$W_a = -[I + WHA]^{-1} W \; * \tag{70}$$

*The W matrix of Fig. 6 and, for example, Eq. 70, is not the W matrix of the regulator equation $U = Wx(0)$.

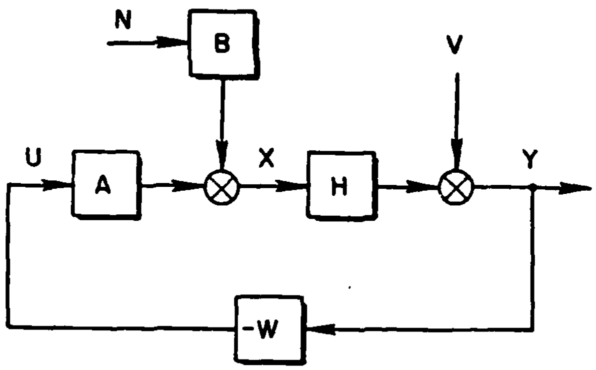

Figure 6. Direct Solution of the Linear Stochastic Optimal Control Problem*

Thus, we can minimize Eq. 66 with respect to W_a and assure closed-loop stability. The W-H equation becomes (using the variational method of Ref. 3 or see Appendix C).

$$[R + A_*QA]W_a[\varphi_{\overline{V}V'} + H\ B\ \varphi_{\overline{N}N'}B_*H_*] \tag{71}$$

$$+ A_*Q\ B\ \varphi_{\overline{N}N'}B_*H_* \quad = \quad \psi$$

for the special case where N and V are independent. Therefore, solve the W-H equation for W_a and compute W using Eq. 70, i.e.,

$$W \quad = \quad -W_a[I + HAW_a]^{-1} \tag{72}$$

The separability principle is still very much in evidence in Eq. 71, since $[R + A_*QA]$ determines that group of closed-loop poles which correspond to the optimal regulator solution, while $\varphi_{\overline{V}V'} + HB\varphi_{\overline{N}N'}(\overline{H}\ \overline{B})'$ determines the

---

*The W matrix of Fig. 6 and, for example, Eq. 70, is not the W matrix of the regulator equation $U = Wx(0)$.

remaining closed-loop poles which correspond to the optimal filter/observer solution. Equation 71 can also be solved using the algebraic methods previously discussed.*

Application of this approach gives the direct relationship of the controller to the output since

$$U = -WY \tag{73}$$

An effect of formulating the problem in this way can be a dramatic reduction in dimensionality of the problem solution. Application to the illustrative example given in the previous subsection yields the exact same transfer function W with less numerical detail (i.e., the solution is the same as Eq. 65.)

*In addition, a spectral factorization algorithm is given in Appendix B.

SECTION IV

THE OPTIMUM COUPLER PROBLEM

The prime objective of the research effort reported herein was to develop an optimal closed-loop solution for direct digital control of continuous plants using a continuous cost function. As noted in the introduction, the solution consists of two parts — an optimal discrete control law and the optimal (continuous) data holds. We have elected to proceed with the development by extending the second method of the previous section in a manner which accounts for sampled output signals which are to be processed by a digital computer and outputted through a data hold (coupler) to the control actuators. In the case of the continuous filter/observer problem two spectral matrices came into play; the matrix which pre-multiplied the unknown of the W-H equation was recognized as the regulator spectral matrix, whereas the post-multiplier was recognized as the filter/observer spectral matrix. Both were, in the illustrative example, rational functions of s. In the W-H equation for the optimum coupler case, it will be seen that the pre-multiplier remains a function of s while the post-multiplier becomes a function of the delay operator $z = e^{sT}$.

The situation of interest is depicted in Fig. 7. For brevity, N + x(0) is taken as N. That is, the initial condition input vector will be suppressed.

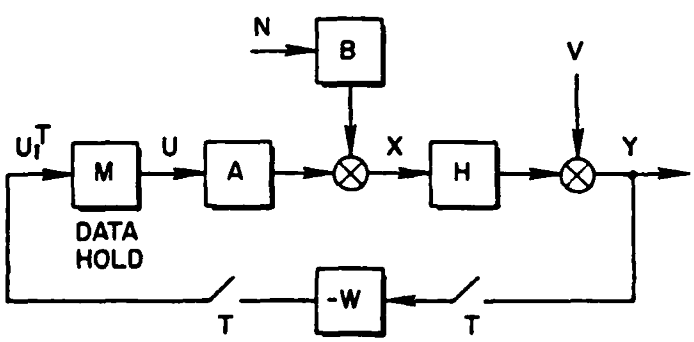

Figure 7. Linear Stochastic Optimal Discrete Control of a Continuous Plant

Let the integrand of the performance index, call it Φ, be the expected value of the usual quadratic index:

$$\widetilde{\Phi} = \widetilde{X_* QX + U_* RU} \tag{74}$$

First develop expressions for the continuous X and U (the superscript T will be used to denote that a signal is impulse sampled at 1/T samples/second; a slash will be used to denote the transpose):

$$U = -MU_1^T = -M[I + W^T(HAM)^T]^{-1}W^T[V + HBN]^T \tag{75}$$

$$X = -AM[I + W^T(HAM)^T]^{-1}W^T[V + HBN]^T + BN \tag{76}$$

We choose to optimize with respect to the matrix of transfer functions between $[V + HBN]^T$ and U^T; therefore, let

$$W_a^T = -[I + W^T(HAM)^T]^{-1} W^T \tag{77}$$

and define

$$\xi_1 \equiv V + HBN \tag{78}$$

$$\xi_2 \equiv BN \tag{79}$$

so that Eqs. 75 and 76 become

$$U = MW_a^T \xi_1^T \tag{80}$$

$$X = AMW_a^T \xi_1^T + \xi_2 \tag{81}$$

Substituting Eqs. 80 and 81 into Eq. 74 gives the integrand of the performance index

$$\Phi = [\xi^T_{1*} W^T_{a*} M_*] \, R[MW_a^T \xi_1^T]$$

$$+ [\xi_{2*} + \xi^T_{1*} W^T_{a*} M_* A_*] \, Q[AMW_a^T \xi_1^T + \xi_2] \tag{82}$$

Next, take the gradient of Eq. 82 with respect to the unknowns MW_a^T, take the expectation and arrive at the W-H equation:

$$[R + A_* QA] MW_a^T \varphi'(\xi_1^T \xi_1^{T'}) + A_* Q\varphi'(\xi_1^T \xi_2) = \psi \tag{83}$$

Assuming V and N to be independent noise processes gives (see Fig. 8):

$$\varphi_{\xi_1^T \xi_1^{T'}} = \frac{1}{T}\left[\varphi_{\overline{V}V'} + HB\varphi_{\overline{N}N'} B_* H_*\right]^T \tag{84}$$

$$= \frac{1}{T}\left[\varphi^T_{\overline{V}V'} + (HB\varphi_{\overline{N}N'} B_* H_*)^T\right] \tag{85}$$

The 1/T scale factor is in keeping with the definitions given in Chapter 10 of Ref. 7.* In a like manner, the cross spectra between the sampled vector ξ_1^T and the continuous vector ξ_2 is simply the scaled continuous spectra (Ref. 7):

$$\varphi_{\overline{\xi_1^T}\xi_2} = \frac{1}{T} B\varphi_{\overline{N}N'} B_* H_* \tag{86}$$

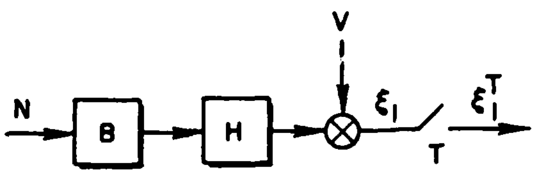

Figure 8. Spectral Noise Model

*The use of 1/T in Ref. 7 appears to be a "matter of convenience." Other reference sources (e.g., Ref. 8) do not use it.

The W-H equation now takes the form

$$[R + A_*QA]\, MW_a^T \left[\frac{1}{T}\,\varphi^T_{\overline{V}\overline{V}'} + \frac{1}{T}\,(HB\varphi_{\overline{N}\overline{N}'}B_*H_*)^T\right]$$

$$+ \frac{1}{T}\,A_*QB\varphi_{\overline{N}\overline{N}'}B_*H_* = \psi \qquad (87)$$

A. SUMMARY OF OPTIMAL COUPLER PROBLEM

Given the digitally controlled continuous system of Fig. 9, the W-H equation, resulting from minimizing the expected value of a quadratic index, is:

$$[R + A_*QA](MW_a^T) \left[\frac{\varphi^T_{\overline{V}\overline{V}'}}{T} + \frac{1}{T}\,(HB\varphi_{\overline{N}\overline{N}'}B_*H_*)^T\right]$$

$$+ \frac{A_*QB\varphi_{\overline{N}\overline{N}'}B_*H_*}{T} = \psi \qquad (88)$$

where W_a^T is defined as

$$W_a^T = -[I + W^T(HAM)^T]^{-1}\,W^T \qquad (89)$$

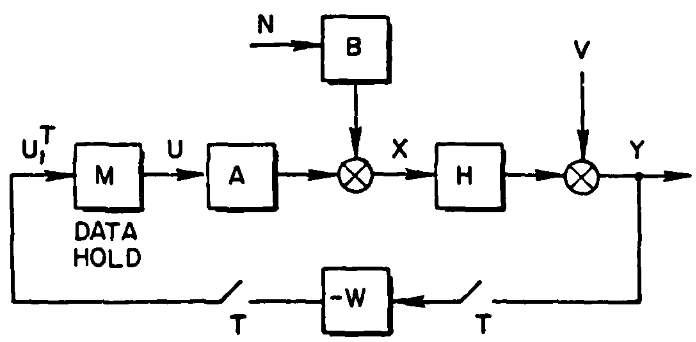

Figure 9. Linear Stochastic Optimal Discrete Control of a Continuous Plant

When Eq. 88 has been solved for MW_a^T, W_a^T is chosen to be the discrete portion and M (the coupler) is taken as the continuous part. A straightforward computation then provides the digital control law

$$W^T = W_a^T[I + (HAM)^T W_a^T]^{-1} \tag{90}$$

which, in turn, yields the equation relating the continuous control output to the sampled measurements:

$$u = -MW^T y^T \tag{91}$$

The W-H equation (Eq. 88) can be solved in two ways: 1) the direct method; or 2) spectral factorization. The first approach requires only the application of the fundamental concept that MW_a^T must be constructed in a manner which forces any pole (be it in the s- or z-domain), which can generate a positive function of time, to cancel into each and every component of ψ. This is an algebraically simple technique. The more computationally difficult spectral factorization solution method is described in Appendix B.

B. OPTIMAL COUPLER — SCALAR EXAMPLE

A scalar, single control point example affords the opportunity to gain familiarity with the mathematical manipulations.

Let

$$A = B = \frac{1}{s+1} \,, \quad H = 1.0 \tag{92}$$

$$R = 1 \,, \quad Q = 8 \,, \quad \varphi_{VV} = 1 \,, \quad \varphi_{NN} = 16 \tag{93}$$

Therefore,

$$R + A_*QA = 1 + \frac{8}{-s^2+1} = \frac{-s^2+9}{-s^2+1} \tag{94}$$

Let

$$GG_* = \frac{\varphi^T_{VV'}}{T} + \frac{1}{T}\left(HB\varphi_{NN'}B_*H_*\right)^T$$

$$= \frac{1}{T^2} + \frac{1}{T}\left(\frac{15}{-s^2+1}\right)^T$$

$$= \frac{1}{T^2} + \frac{15}{2T}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]^T \tag{95}$$

The power spectra of sampled white noise has been taken as $1/T^2$ rather than $1/T$ (see Ref. 7 for a discussion). Continuing, we find

$$GG_* = \frac{1}{T^2} + \frac{15}{2T}\left[\frac{z}{z-e^{-T}} - \frac{z}{z-e^{T}}\right]$$

$$= \frac{1}{T^2}\left[1 + \frac{\frac{15T}{2}\, z(e^{-T}-e^{T})}{(z-e^{-T})(z-e^{T})}\right] \tag{96}$$

$$= \frac{1}{T^2}\left[\frac{z^2 - \left\{e^{T} + e^{-T} - \frac{15}{2}\, T(e^{-T}-e^{T})\right\}z + 1}{(z-e^{-T})(z-e^{T})}\right]$$

$$= \frac{1}{T^2}\,\frac{(z-e^{-BT})(z-e^{BT})}{(z-e^{-T})(z-e^{T})} \tag{97}$$

Thus, the optimal filter/observer pole is defined by $z - e^{-BT}$; the value of B as a function of T is given in Table 1

TABLE 1

T	B
0	4.
0.01	3.9998
0.1	3.9769
1.	3.0285

The only remaining computation is

$$\frac{1}{T} A_* Q B \Phi_{\overline{NN}'} B_* H_* = \frac{120}{T(-s+1)^2(s+1)} \tag{98}$$

The W-H equation is therefore

$$\frac{(-s+3)(s+3)}{(-s+1)(s+1)} M W_a^T \frac{(z-e^{-BT})(z-e^{BT})}{T^2(z-e^{-T})(z-e^{T})}$$

$$+ \frac{120}{T(-s+1)^2(s+1)} = \psi \tag{99}$$

The application of the direct approach requires that MW_a^T be such that the numerator of ψ cancels all those poles which can produce positive time functions — that is, cancel the terms s + 1 and $z - e^{-T}$. A selection of

$$MW_a^T = \frac{a_0(z - e^{-T})}{(s+3)(z - e^{-BT})} , \tag{100}$$

where a_0 is an undetermined coefficient, is sufficient to achieve this goal. Substitution of Eq. 100 into Eq. 99 gives

$$\frac{(-s+3)(s+3)}{(-s+1)(s+1)} \times \frac{a_0(z-e^{-T})}{(s+3)(z-e^{-BT})}$$

$$\times \frac{(z-e^{-BT})(z-e^{BT})}{(z-e^{-T})(z-e^{T})} + \frac{120T}{(-s+1)^2(s+1)} = \psi \quad . \tag{101}$$

or

$$\frac{(-s+1)(-s+3)a_0(z-e^{BT}) + 120T(z-e^{T})}{(-s+1)^2(s+1)(z-e^{T})} = \psi \tag{102}$$

In Eq. 102, a_0 must be selected so that the numerator is zero when $s = -1$; therefore,

$$a_0 = -\frac{120T(e^{-T}-e^{T})}{2(4)(e^{-T}-e^{BT})} = -15T\,\frac{(1-e^{2T})}{[1-e^{(1+B)T}]} \tag{103}$$

For example, when $T = 1$ sec,

$$MW_a^T = \frac{-1.736929333(z-e^{-T})}{(s+3)(z-e^{-BT})} \quad , \qquad T = 1.0 \tag{104}$$

Notice the general result. The data-hold poles are defined by the (continuous) regulator spectral matrix, while the "discrete" poles of W_a^T are determined by the sampled filter/observer spectral matrix. Letting

$$M = \frac{1}{s+3} \quad , \quad W_a^T = \frac{a_0(z-e^{-T})}{(z-e^{-BT})} \tag{105}$$

one then proceeds to compute the digital control law using the equation

$$W^T = -W_a^T[I + (HAM)^T W_a^T]^{-1} \tag{106}$$

First,

$$(HAM)^T = \left[\frac{1}{(s+1)(s+3)}\right]^T = \left[\frac{1/2}{s+1} - \frac{1/2}{s+3}\right]^T$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{-T}} - \frac{z}{z-e^{-3T}}\right]$$

$$= \frac{1}{2}\,\frac{(e^{-T}-e^{-3T})z}{(z-e^{-T})(z-e^{-3T})} \tag{107}$$

Therefore,

$$(HAM)^T W_a^T = \frac{1}{2}\,\frac{(e^{-T}-e^{-3T})z}{\cancel{(z-e^{-T})}(z-e^{-3T})}\,\frac{a_0\cancel{(z-e^{-T})}}{(z-e^{-BT})}$$

and

$$I + (HAM)^T W_a^T$$

$$= \frac{z^2 - \left\{e^{-3T} + e^{-BT} - \frac{a_0}{2}(e^{-T}-e^{-3T})\right\}z + e^{-(3+B)T}}{(z-e^{-3T})(z-e^{-BT})}$$

$$\equiv \frac{(z-e^{-\alpha T})(z-e^{-T})}{(z-e^{-3T})(z-e^{-BT})} \quad , \quad \alpha = 2 + B \tag{108}$$

That is, as a consequence of the W-H process, the numerator of $I + (HAM)^T W_a^T$ contains the "open-loop root," $z - e^{-T}$, as an exact factor. Therefore,

$$W^T = \frac{-a_0(z-e^{-T})}{z-e^{-BT}} \, \frac{(z-e^{-3T})(z-e^{-BT})}{(z-e^{-\alpha T})(z-e^{-T})}$$

$$= \frac{-a_0(z-e^{-3T})}{(z-e^{-\alpha T})} \tag{109}$$

It is interesting to note that the output equation

$$U = -MW^T Y^T \tag{110}$$

can be written as

$$U = -\left[\frac{1}{s+3}\right]\left[\frac{-a_0(z-e^{-3T})}{z-e^{-\alpha T}}\right] Y^T$$

$$= -\underbrace{\left[\frac{1-e^{-T(s+3)}}{s+3}\right]}_{M}\underbrace{\left[\frac{-a_0 z}{z-e^{-\alpha T}}\right]}_{W^T} Y^T$$

$$= -MW^T Y^T \tag{111}$$

That is, the data hold can be viewed in a manner quite similar to the zero order hold (see Fig. 10). It is now apparent that the optimal coupler solution defines a data hold which forces the plant to follow a path, during the intersample period, that is in a sense "scheduled" by the continuous constraints placed on the solution by the (continuous) spectral matrix $R + A_* QA$.

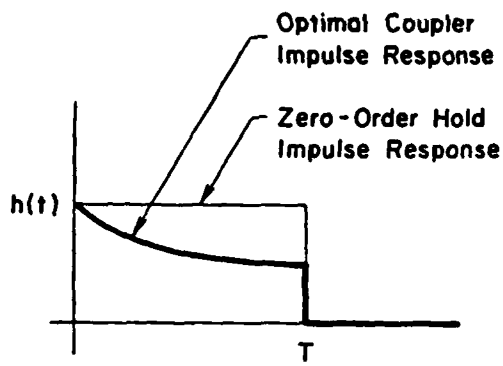

Figure 10. Impulse Response of Zero-Order Hold and Optimal Coupler

The example has focused on the numerical details of solving the W-H equation when the regulator spectral matrix is a function of s and the filter/observer spectral matrix is a function of z. In the next section we treat a "limiting form" problem in order to focus on the properties of the optimal coupler (in distinction to W-H solution techniques).

C. THE OPTIMAL REGULATOR/COUPLER PROBLEM

We return now to the notation of Section II and treat the optimal coupler for the special case where N and V are zero — the only excitation being the initial conditions. Recall, for the optimal regulator, the Wiener-Hopf approach solves for the optimal controller. Therefore, it does not distinguish between the two configurations shown in Fig. 11. Given a performance index for which the integrand is

$$\Phi = X_* QX + U_* RU \tag{112}$$

the W-H equation is

$$[R + A_* QA]U_o + A_* QBx(0) = \psi \tag{113}$$

and the solution is

$$U_o = W(s)x(0) \tag{114}$$

For the configuration of Fig. 11a, the compensation is

$$K = -W[B + AW]^{-1} \tag{115}$$

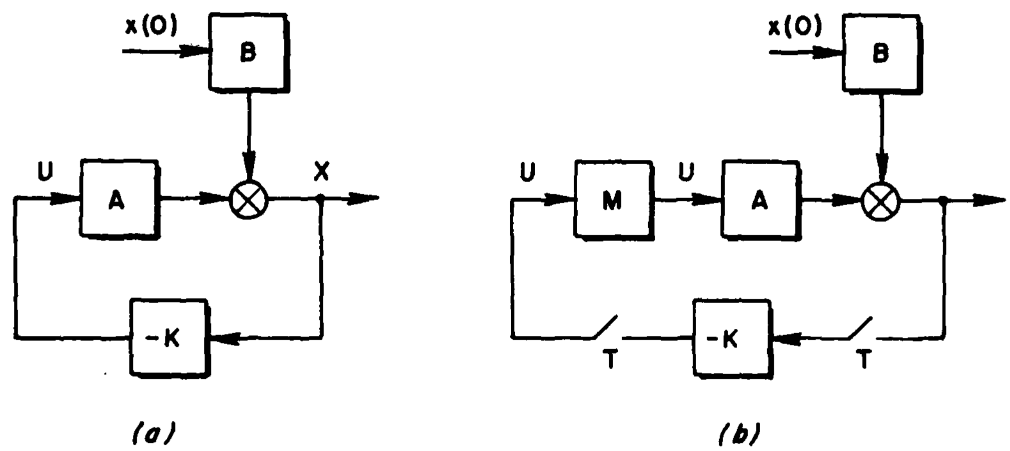

Figure 11. Analog vs. Digital Controller

Whereas for Fig. 11b the compensation and data holds are defined by:

$$MK^T = -W\{[B + AW]^T\}^{-1} \tag{116}$$

If a discrete index is used,

$$\Phi = X_*^T QX^T + U_*^T R U^T \tag{117}$$

then the W-H equation is (M given)

$$[R + (AM)_*^T Q(AM)^T] U^T + (AM)_*^T QB^T x(0) = \psi \tag{118}$$

The solution is

$$U^T = W^T x(0) \tag{119}$$

The gains can be found with the (z-domain) initial value theorem.

Clearly, for a problem formulation which is as simple as the regulator, the use of a continuous index forces the same smooth continuous motions in the state and controller deflections for the discretely controlled case as it does for the analog controller. The situation is not nearly as simple to predict when the discrete index is used, because the migration of z-plane "zeros, as a function of q and r, is usually not very transparent.

We will solve the W-H equations and give transient responses to demonstrate these points. Specifically, three cases are considered:

1) Continuous index, continuous controller.
2) Continuous index, discrete controller, data holds specified by W-H solution.
3) Discrete index, discrete controller, data holds specified a priori.

OPEN-LOOP PLANT

Let

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 5 \end{bmatrix} U \qquad (120)$$

so that

$$X(s) = \frac{\begin{bmatrix} s+1 \\ 5s+1 \end{bmatrix}}{s(s-3)} U + \frac{\begin{bmatrix} s-4 & 1 \\ -4 & s+1 \end{bmatrix}}{s(s-3)} x(0) = AU + B\,x(0) \qquad (121)$$

Suppose

$$R = 1 \text{ (a scalar)} \quad , \quad Q = \begin{bmatrix} \frac{28}{75} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \qquad (122)$$

$$R + A_{*}QA \equiv \frac{(s^2+1.2s+.2)(s^2-1.2s-.2)}{s(s-3)(-s)(-s-3)} = \frac{\Delta\bar{\Delta}}{D\bar{D}} \qquad (123)$$

Since there is only one controller, exercising the first W-H condition gives

$$U = W(s)\,x(0) = \frac{[.8(s+1) \mid -(s+.2)]}{s^2+1.2s+.2} x(0)$$

$$= \left[\frac{.8}{s+.2} \;\middle|\; \frac{-1}{s+1} \right] x(0) \qquad (124)$$

The matrix of closed-loop transfer functions is:

$$X = [B + AW]x(0) = \left\{ \frac{\begin{bmatrix} s-4 & 1 \\ -4 & s+1 \end{bmatrix}}{s(s-3)} + \frac{\begin{bmatrix} s+1 \\ 5s+1 \end{bmatrix}}{s(s-3)} \left[\frac{.8}{s+.2} \;\vdots\; \frac{-1}{s+1} \right] x(0) \right\}$$

$$\equiv \begin{bmatrix} \frac{1}{s+.2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} x(0) \tag{125}$$

The feedback gains can be computed using either the initial value theorem or Eq. 115:

$$K = [\, -.8 \;\vdots\; 1 \,] \tag{126}$$

As noted previously, Case 2 must have the same W-H equation, and therefore the same solution:

$$U_o = W(s)\, x(0) = \left[\frac{.8}{s+.2} \;\vdots\; \frac{-1}{s+1} \right] x(0) \tag{127}$$

The compensation (data holds and gains) is defined by Eq. 116:

$$MK^T = -W\left\{[B + AW]^T\right\}^{-1} = -\left[\frac{.8}{s+.2} \;\vdots\; \frac{-1}{s+1} \right] \begin{bmatrix} \frac{z}{z-e^{-.2T}} & 0 \\ 0 & \frac{z}{z-e^{-T}} \end{bmatrix}$$

$$\equiv \left[\frac{-.8[1-e^{-T(s+.2)}]}{s+.2} \;\vdots\; \frac{(1-e^{-T(s+1)})}{s+1} \right] \tag{128}$$

That is,

$$M = \left[\frac{1-e^{-T(s+.2)}}{s+.2} \;\vdots\; \frac{1-e^{-T(s+1)}}{s+1} \right] , \quad K^T = \begin{bmatrix} -.8 & 0 \\ 0 & 1 \end{bmatrix} \qquad (129)$$

One may verify, by direct substitution of Eq. 127, that

$$U = -M[I + K^T(AM)^T]^{-1} K^T B^T x(0) \equiv W(s)\,x(0)$$

and therefore

$$X = (B + AW)\,x(0)$$

Next, it is a straightforward exercise in discrete regulator theory to obtain the Case 3 result [q's and r's were used which forced $(z - e^{-T})(z - e^{-.2T})$ as the closed-loop poles]. The gains that achieve this, as a function of the frame time T, are given in Table 2 ($u = -KX^T = -[K_1 \quad K_2][X_1 \quad X_2]^T$).

TABLE 2

	T = .1	T = .5	T = 1
K_1	−.792556055	−.771538772	−.758184683
K_2	.954136595	.836065309	.776195811

The discrete controller deflections, as a function of x(0), can be computed using

$$U^T = -[I + K(AM)^T]^{-1} K B^T x(0) \qquad (132)$$

and the continuous state motions using

$$X = Bx(0) + AMU^T \qquad (133)$$

(In computing Eq. 133 it is best to carry the analysis to a point where the strongly unstable open-loop root has been cancelled out.) The transient responses, for $T = 1$, $x_1(0) = 1$, $x_2(0) = 4$ are shown in Fig. 12.

Note the poor transient response of the system designed with discrete regulator theory. Whereas the continuous design was "decoupled" as far as the initial condition response (each state containing only one mode), the discrete design forced the correct poles, but the zeros go where they will. This can be verified by checking the X^T closed-loop transfer functions:

$$X^T = Z \frac{\begin{bmatrix} z-.36991217 & .0365270 \\ .024976780 & z-.81669801 \end{bmatrix}}{(z-e^{-.2T})(z-e^{-T})} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{134}$$

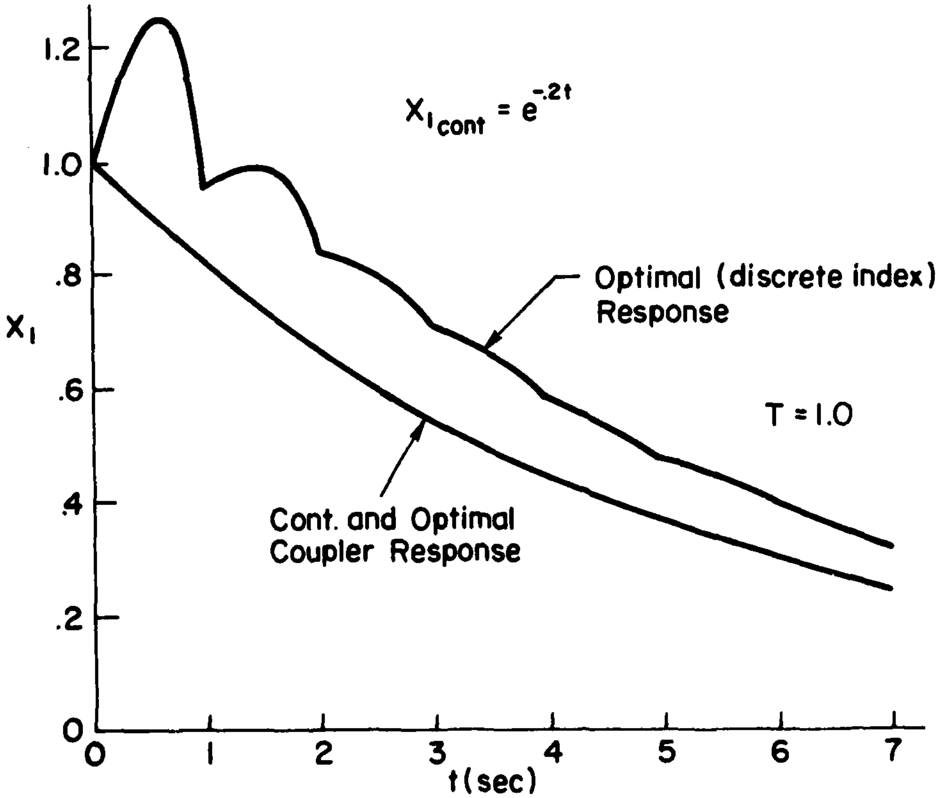

Figure 12. Comparison of Transient Responses

To summarize, the optimal coupler solution forces the same initial condition response as associated with the continuous regulator design. The discret regulator design exhibits undesirable inter-sample response. Indeed, with only two gains, the closed loop poles can be specified but the zeroes go where they will — resulting in a design which cannot even match the continuous response at the sampling instants.

SECTION V

SUMMARY

A new class of problem is solved using the W-H approach. The problem treats the case of a continuous performance index, together with a continuous plant and sampled measurements to specify both the optimal (discrete) digital control law and the optimal (continuous) data holds to be used for coupling the digital control law to the plant actuators. The optimal coupler orchestrates an inter-sample controller response leading to closed loop systems which exhibit characteristics markedly superior to their discrete regulator counterparts.

There are several important circumstances in which direct digital design with respect to a continuous cost function is crucial to efficient and successful synthesis of digital control laws. These circumstances are:

- Existing systems for which the limits on core, word length, and computational speed have been reached (for example, V/STOLAND).
- Applications where outer loops are updated at inherently slow rates [for example, SIN-42, MLS, NASA's remotely piloted research vehicle (RPRV)].
- Applications wherein the cost of a fast minicomputer is not justifiable but a microprocessor is cost effective (for example, general aviation, digital controllers for missiles, RPV's and other smart, expendable weapons).
- Application of low data rate techniques for real-time simulation problems.

In addition, the developments of Sections II and III make clear the uniform manner in which the direct solution method treats singular problems. That is, no modification is required to solve cases for which some (or all) of the feedback regulator gains or some (or all) of the feedforward gains of the filter/observer go to infinity.

Furthermore, the examples of Appendix A demonstrate:

- The R weighting matrix need not be positive definite nor is it necessary that R^{-1} exists. Q can be ≤ 0.
- Singular regulator and filter/observer problems (e.g., the cheap control problem) can be handled neatly.
- The weighting matrices, R and Q, can be explicit functions of frequency.
- There are advantages in using non-diagonal Q and R matrices.

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APPENDIX A

ILLUSTRATIVE REGULATOR EXAMPLES

Illustrative regulator examples, designed to demonstrate the "bulleted" items of the abstract, are synopsized in this appendix. In general, we list:

- The open loop plant; $\dot{x} = Fx + Gu$
- The transform of the open-loop plant; $x = AU + Bx(0)$
- The weighing matrices; R, Q
- Wiener-Hopf equation and the assumed form for the optimal control
- First W-H condition
- Second W-H condition
- Solution for the unknowns of U_o
- Optimal control gains
- Matrix of closed-loop transfer functions

Observation

Choice of Q, R excludes feedback from second state

Open-Loop Plant

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix} U \qquad X(s) = \frac{\begin{bmatrix} s-1 & 2s+7 \\ -4s-2 & s-4 \end{bmatrix}}{(s+1)(s+2)} U + \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{D(s)} x(0) = AU + Bx(0)$$

Weighting Matrices

$$R = \begin{bmatrix} 7/8 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 23/9 & -7/9 \\ -7/9 & 0 \end{bmatrix} \qquad \det[R + A_* QA] = \Delta\tilde{\Delta}$$

$$= (s+3)(s+4)(-s+3)(-s+4)$$

W-H Equation

$$[R + A_* QA]U_o + A_* QB\, x(0) = \psi \qquad\qquad U_o = \frac{\begin{bmatrix} a_0 s + a_1 \\ b_0 s + b_1 \end{bmatrix}}{\Delta}$$

$$U_o = Wx(0)$$

1st W-H Condition

$$(s-1)(a_0 s + a_1) + (2s+7)(b_0 s + b_1) = -\tilde{\Delta}[s+3 \;\vdots\; 1]\, x(0) \qquad s = -1, -2$$

2nd W-H Condition

$$\left(.875 s^4 - \frac{329}{24} s^2 + 3.5\right)(a_0 s + a_1) + \left(\frac{-35}{3} s^2 - 49s - 14\right)(b_0 s + b_1) = 0 \qquad s = -3, -4$$

Solution for Unknowns

$$\begin{array}{l} s=-2 \\ s=-1 \\ s=-3 \\ s=-4 \end{array} \begin{bmatrix} 6 & -3 & -6 & 3 \\ 2 & -2 & -5 & 5 \\ 21 & -7 & -12 & 4 \\ 28 & -7 & -16 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -12 & -6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -8/9 & 0 \\ -24/9 & -8/9 \\ -14/9 & 0 \\ -42/9 & -14/9 \end{bmatrix}$$

Optimal Control Gains

$$U_o = W(s)\, x(0) = \frac{\left[\begin{array}{c|c} -(8/9)s - 24/9 & -8/9 \\ \hline -(14/9)s - 42/9 & -14/9 \end{array}\right]}{(s+3)(s+4)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Therefore,

$$K = \lim_{s \to \infty} -sW(s) = \begin{bmatrix} 8/9 & 0 \\ 14/9 & 0 \end{bmatrix}$$

Matrix of Closed-Loop Transfer Functions

$$X = [B + AW]x(0) = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} - \frac{\begin{bmatrix} s-1 & 2s+7 \\ -4s-2 & s-4 \end{bmatrix}}{(s+1)(s+2)} \frac{\begin{bmatrix} 8s+24 & 8 \\ 14s+42 & 14 \end{bmatrix}}{9(s+3)(s+4)} = \frac{\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}}{(s+3)(s+4)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

EXAMPLE — UNSTABLE OPEN-LOOP PLANT, SINGLE CONTROLLER

Observations

W-H is "dimensioned" by the controller, not the state; the single controller problem requires only the first W-H condition.

We can keep an open-loop root as a closed-loop root if we so desire.

Q can be identically zero; the method continues to work without the need for modification.

Open-Loop Plant

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} U \qquad X(s) = \frac{\begin{bmatrix} -2s+1 \\ s(-2s+1) \\ s^2-s-1 \end{bmatrix}}{s(s^2-2s+2)} U(s) + \frac{\begin{bmatrix} (s-1)^2 & s-1 & -1 \\ -s & s(s-1) & -s \\ s-1 & 1 & s^2-s+1 \end{bmatrix}}{s(s^2-2s+2)} x(0)$$

Weighting Matrices

$$Q \equiv 0 \qquad \det[R + A_* Q A] \equiv R D\bar{D} = R[s(s^2-2s+2)][-s(s^2+2s+2)]$$

$$R > 0 \qquad \Delta = s(s^2+2s+2)$$

$$[R + A_* Q A]U_o + A_* Q B x(0) = \psi \qquad U_o = W x(0) \qquad U_o = \frac{a_0 s^2 + a_1 s + a_2}{s(s^2+2s+2)}$$

1st W-H Condition (only one needed in the single controller case)

$$(-2s+1)(a_0 s^2 + a_1 s + a_2) = -\Delta[(s-1)^2 \quad s-1 \quad -1]x(0) \qquad s = 0,\ 1+j$$

Solutions for Unknowns

when $D = 0$ $\quad a_2 \equiv 0$

$$\begin{bmatrix} 1 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & 12 & 4 \\ -16 & -8 & -16 \end{bmatrix}$$

Optimal Control Gains

$$U_o = W(s)\,x(0) = \frac{1}{5}\left[\frac{4s-16 \ \vdots\ 12s-8 \ \vdots\ 4s-16}{s^2+2s+2}\right]x(0) \qquad K = \lim_{s \to \infty} -sW(s)$$

$$= \left[-\frac{4}{5} \quad -\frac{12}{5} \quad -\frac{4}{5}\right]$$

Matrix of Closed-Loop Transfer Functions

$$X = [B + AW]x(0) = \left\{ \frac{\begin{bmatrix} (s-1)^2 & s-1 & -1 \\ -s & s(s-1) & -s \\ s-1 & 1 & s^2-s+1 \end{bmatrix}}{s(s^2-2s+2)} + \frac{\begin{bmatrix} -2s+1 \\ -2s^2+s \\ s^2-s-1 \end{bmatrix}}{D} \, \frac{[4s-16 \quad 12s-8 \quad 4s-16]}{5(s^2+2s+2)} \right\} x(0)$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \equiv \frac{\begin{bmatrix} 5s^2+10s-3 & 5s-9 & -13 \\ -13s & 5s^2-9s & -13s \\ 9s+3 & 12s+9 & 5s^2+19s+13 \end{bmatrix}}{5s(s^2+2s+2)} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

Note: If the open-loop plant had been stable, the selection of Q = 0 would result in a feedback gain matrix K ≡ 0.

EXAMPLE — Q, R MAY BE FUNCTIONS OF FREQUENCY

Observation

Lead/lag compensation

Open-Loop Plant

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{s(s+1)} U(s) + \frac{\begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}}{s(s+1)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Weighting Matrices

$$R = \frac{-s^2 + 5112}{-s^2 + 9} \qquad Q = \begin{bmatrix} 97344 & 0 \\ 0 & -4836 \end{bmatrix} \qquad \det[R + A_* QA] = (s+4)(s^2 + 27s + 234) \times (-s+4)(s^2 - 27s + 234)$$

W-H Equation

$$[R + A_* QA]U_0 + A_* QBx(0) = \psi \;; \quad \frac{\Delta\bar{\Delta}(a_0 s^2 + a_1 s + a_2) + \Delta(s^2+9)[97344(s+1) \;\vdots\; 4836 s^2 + 97344]}{D\bar{D}\Delta(s+3)(-s+3)} x(0) =$$

W-H Requirements

$$\Delta = (s+4)(s^2 + 27s + 234) \qquad U_0 = \frac{a_0 s^2 + a_1 s + a_2}{\Delta}$$

ψ Numerator $\equiv 0$ when $D\Delta = 0,\ s = -3$

Solutions for Unknowns

$s = 0$

$a_2 = [-936 \quad -936]\,x(0)$

$s = -1$

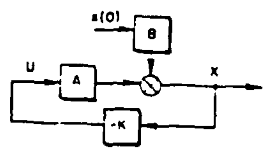
$9a_0 - 3a_1 + a_2 = 0$

$s = -3$

$a_0 - a_1 + a_2 = [0 \quad -624]\,x(0)$

$$\begin{bmatrix} 9 & -3 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -624 \\ -936 & -936 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -312 & 0 \\ -1248 & -312 \\ -936 & -936 \end{bmatrix}$$

Optimal Control Compensation



$$U = Wx(0) = \frac{[-(312s^2 + 1248s + 936) \;\vdots\; -(312s + 936)]}{(s+4)(s^2 + 27s + 234)} x(0)$$

$$= \frac{-[312(s+3)(s+1) \;\vdots\; 312(s+3)]}{(s+4)(s^2 + 27s + 234)} x(0)$$

$$K = -W[B + AW]^{-1} = \begin{bmatrix} \frac{312(s+3)}{s+30} & \vdots & 0 \end{bmatrix}$$

Matrix of Closed-Loop Transfer Functions

$$X = [B + AW]x(0) = \frac{\begin{bmatrix} s^2 + 31s + 30 & s + 30 \\ -312(s+3) & s(s+30) \end{bmatrix}}{(s+4)(s^2 + 27s + 234)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

EXAMPLE — ONE "CHEAP" CONTROL

Open-Loop Plant

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ -4 & 2 \\ 1 & 0 \end{bmatrix} U \qquad X(s) = \frac{\begin{bmatrix} -4s+1 & (s+1)^2 \\ s(-4s+1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)} U + \frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)} x(0)$$

$$= Au + Bx(0)$$

Weighting Matrices

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 30 \end{bmatrix} \qquad Q = \begin{bmatrix} 94 & & \\ & 0 & \\ & & -705 \end{bmatrix} \qquad \det[R + A_* QA] \;\propto\; s^2 + 2s + 2 \;=\; (s+1)^2 + (1)^2$$

W-H Equation

$$[R + A_* QA]U_o + A_* QBx(0) = \psi \qquad U_o = Wx(0) \qquad U_o = \frac{\begin{bmatrix} a_0 s^2 + a_1 s + a_2 \\ b_1 s + b_2 \end{bmatrix}}{s^2 + 2s + 2}$$

Note: Place the "burden" of the extra coefficient on the controller which is excluded from the performance index.

1st W-H Condition

$$(-4s+1)(a_0 s^2 + a_1 s + a_2) + (s^2 + 2s + 1)(b_1 s + b_2) = -\Delta[s^2 - 1 \quad s+1 \quad 5]x(0) \qquad s = 0, +2j$$

2nd W-H Condition

$$(705s^4 + 2209s^2 + 611)(a_0 s^2 + a_1 s + a_2) + (329s^3 + 564s^2 + 846s + 611)(b_1 s + b_2) = 0 \qquad s = -1 + j$$

Solution for Unknowns

$$\begin{bmatrix} -4 & 16 & 1 & -8 & -3 \\ 32 & 2 & -8 & -6 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ -8836 & 6627 & -2209 & -799 & 423 \\ 4418 & 2209 & -4418 & 47 & 376 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -10 & 10 & 10 \\ 20 & 0 & -20 \\ 2 & -2 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0) \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \frac{\begin{bmatrix} -136 & -18 & -327 \\ -425 & -60 & -750 \\ -289 & -87 & -858 \\ -799 & -282 & -1128 \\ 799 & -423 & -1692 \end{bmatrix}}{255} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

Optimal Control Gains

$$U = W(s)\,x(0)$$

$$= \frac{1}{255} \frac{\left[\begin{array}{c|c|c} -136s^2 - 425s - 289 & -18s^2 - 60s - 87 & -327s^2 - 750s - 858 \\ \hdashline -799s + 799 & -282s - 423 & -1128s - 1692 \end{array}\right]}{(s^2 + 2s + 2)} x(0) \quad ; \quad K = \lim_{k_o \to \infty} \frac{\begin{bmatrix} 136k_o & 18k_o & 327k_o \\ 799 & 282 & 1128 \end{bmatrix}}{255}$$

Note: The first controller gains go to infinity, at a rate determined by the ratios of the entries of the first row of the equation for K, above, right.

Matrix of Closed-Loop Transfer Functions

$$X = [B + AW]x(0) = \frac{\begin{bmatrix} 255(s+1) & 45 & 180 \\ 544s - 1309 & 327s + 423 & 1308s + 1692 \\ -136s - 34 & -18s - 42 & -72s - 168 \end{bmatrix}}{255(s^2 + 2s + 2)} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

EXAMPLE — $R \equiv 0$, TWO "CHEAP" CONTROLLERS

Observation

W-H approach requires no modification when $R \equiv 0$.

Open-Loop Plant

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ -4 & 2 \\ 1 & 0 \end{bmatrix} U \qquad X(s) = \frac{\begin{bmatrix} -4s+1 & (s+1)^2 \\ s(-4s-1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)} U + \frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)} x(0)$$

$$= Au + Bx(0)$$

Weighting Matrices

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \det[R + A_* QA] \propto (s+1) \qquad \Delta \text{ reduced by two}$$

W-H Equation

$$[R + A_* QA]U_o + A_* QBx(0) = \psi \qquad U_o = Wx(0) = \frac{\begin{bmatrix} a_1 s + a_2 \\ b_1 s + b_2 \end{bmatrix}}{\Delta}$$

1st W-H Condition

$$(-4s+1)(a_1 s + a_2) + (s+1)^2(b_1 s + b_2) = -\Delta[s^2 - 1 \,\vdots\, s+1 \,\vdots\, 5]x(0) \qquad \text{when } s = 0, +2j$$

2nd W-H Condition

$$N_A(a_1 s + a_2) + N_B(b_1 s + b_2) = 0 \quad \text{when } s = -1$$

where

$$N_A = q_1(-16s^2 + 1) + q_2(-s^2)(-16s^2 + 1) + q_3(s^4 + s^2 + 1)$$

$$N_B = q_1(4s^3 + 9s^2 + 6s + 1) + q_2(-8s^4 + 10s^3 + 3s^2) + q_3(s^3 + 2s^2 + 2s + 1)$$

Solution for Unknowns

$$\begin{bmatrix} 16 & 1 & -8 & -3 \\ 2 & -8 & -6 & 4 \\ 0 & 1 & 0 & 1 \\ 7 & -7 & -3 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 5 & 3 & -5 \\ 10 & -4 & -10 \\ 1 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} x(0) \qquad \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \frac{\begin{bmatrix} -18 & 9 & 10 \\ 12 & -6 & -50 \\ -56 & 15 & 60 \\ 14 & -20 & -80 \end{bmatrix}}{26} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

Optimal Control Gains

$$U = W(s)\,x(0) = \frac{1}{26} \frac{\begin{bmatrix} -18s+12 \,\vdots\, 9s-6 \,\vdots\, 10s-50 \\ -56s+14 \,\vdots\, 15s-20 \,\vdots\, 60s-80 \end{bmatrix}}{(s+1)} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} \qquad K = \lim_{s \to \infty} -sW(s) = \frac{1}{26} \begin{bmatrix} 18s & -9s & -10s \\ 56s & -15s & -60s \end{bmatrix}_{\lim\, s \to \infty}$$

Even though the gains go to infinity, they must do so with the ratios defined by the equation for K and the result will be a well-defined matrix of closed-loop transfer functions. We can, of course, compute the result directly using $X = [B + AW]\,x(0)$.

Matrix of Closed-Loop Transfer Functions

$$X = \left\{ \frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)} + \frac{\begin{bmatrix} -4s+1 & (s+1)^2 \\ s(-4s+1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)} \; \frac{\begin{bmatrix} -18s+12 \,\vdots\, 9s-6 \,\vdots\, 10s-50 \\ -56s+14 \,\vdots\, 15s-20 \,\vdots\, 60s-80 \end{bmatrix}}{26(s+1)} \right\} x(0)$$

$$= \frac{\begin{bmatrix} -30 & 15 & 60 \\ -40 & 20 & 80 \\ -18 & 9 & 36 \end{bmatrix}}{(s+1)} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

The "reduced state" answer is obtained without the necessity of any limiting procedure applied to the feedback gains.

Observation

W-H approach requires no modification when the model is unstable.

$$y = Hx$$

$$\dot{x} = Fx + Gu \Rightarrow X(s) = (Is - F)^{-1}GU + (Is - F)^{-1}x(0) = AU + Bx(0)$$

$$\int_0^\infty \left\{ [\dot{y} - Ly]'Q[\dot{y} - Ly] + u'Ru \right\} dt = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \left\{ [(Is - L)Y]_* Q[(Is - L)Y] + U_* RU \right\} ds$$

W-H Equation

$$[R + A_* H_* (Is - L)_* Q(Is - L)HA]U + A_* H_* (Is - L)_* Q(Is - L)HBx(0) = \psi$$

Example 1

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad H = I \Rightarrow X(s) = \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{(s+1)(s+2)} U(s) + \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

$$R = 0 \qquad Q = I \qquad L = \begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix} \qquad [Is - L] = \begin{bmatrix} s & -1 \\ -12 & s+1 \end{bmatrix} \qquad \det[Is - L] = s^2 + s - 12 = (s+4)(s-3)$$

W-H Equation:

$$\frac{(s^2 - s - 12)(s^2 + s - 12)}{D\bar{D}} \frac{[a_0 s + a_1]}{(s+4)(s-3)} + \frac{(s^2 - s - 12)[-14s - 38 \ \vdots\ s^2 + s - 12]}{D\bar{D}} x(0) = \psi$$

$$U_0 = Wx(0) = \frac{[14s + 38 \ \vdots\ 2s + 14]}{(s+4)(s-3)} x(0) \qquad \text{Therefore,} \quad K = [-14 \ , \ -2]$$

$$X = [B + AW]x(0) = \frac{\begin{bmatrix} s+1 & 1 \\ 12 & s \end{bmatrix}}{(s+4)(s-3)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \equiv [Is - L]^{-1} x(0) \qquad \underline{\text{Check}}$$

Example 2

Same as Example 1, except $R = 0 \quad Q = I \quad L = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

Therefore,

$$[Is - L] = \begin{bmatrix} s-2 & 1 \\ 0 & s-1 \end{bmatrix} \qquad \det[Is - L] = (s-1)(s-2)$$

$$U_0 = Wx(0) = \frac{[-12 \quad 6s]}{s^2 - 3s + 2} x(0) \Rightarrow K = [0 \quad -6]$$

$$X = [B + AW]x(0) = \frac{\begin{bmatrix} s-3 & 1 \\ -2 & s \end{bmatrix}}{s^2 - 3s + 2} x(0) \qquad [Is - L]^{-1} = \frac{\begin{bmatrix} s-1 & -1 \\ 0 & s-2 \end{bmatrix}}{s^2 - 3s + 2} \neq B + AW$$

Therefore,

Choice of $Q = I$ matches only the poles.

APPENDIX B

SOLUTION USING SPECTRAL FACTORIZATION

The regulator W-H equation is

$$[R + A_*QA]U + A_*QBx(0) = \psi \tag{B-1}$$

Since we may write

$$[R + A_*QA] = [I + A_*K']R[I + KA] = F_*RF \tag{B-2}$$

one may verify, by direct substitution, that

$$U_o = -F^{-1} R^{-1} [F_*^{-1} A_*QBx(0)]_+ \tag{B-3}$$

is a solution to Eq. B-1. This solution requires only that R^{-1} exist.

The Wiener-Hopf equation for the filter-observer problem is

$$[R + A_*QA]W_a[\varphi_{\overline{V}V'} + HB\varphi_{\overline{N}N'}B_*H_*] + A_*QB\varphi_{NN'}B_*H_* = \psi \tag{B-4}$$

Setting

$$\varphi_{\overline{V}V'} + HB\varphi_{\overline{N}N'}B_*H_* = GG_* \tag{B-5}$$

$$R + A_*QA = F_*RF \tag{B-6}$$

$$A_*QB\varphi_{\overline{N}N'}B_*H_* = N \tag{B-7}$$

gives the following form for Eq. B-4:

$$(F_*RF)W_aGG_* + N = \psi \tag{B-8}$$

By direct substitution, one may verify

$$W_a = -F^{-1} R^{-1} [F_*^{-1} N G_*^{-1}]_+ G^{-1} \tag{B-9}$$

satisfies Eq. B-8. That is,

$$-F_*[F_*^{-1} N G_*^{-1}]_+ G_* + F_* F_*^{-1} N G_*^{-1} G_* = \psi \tag{B-10}$$

$$F_*\left\{-[F_*^{-1} N G_*^{-1}]_+ + F_*^{-1} N G_*^{-1}\right\} G_* = F_*\left\{F_*^{-1} N G_*^{-1}\right\}_- G_* \tag{B-11}$$

Equation B-11 satisfies the W-H requirement of forcing ψ to consist of time functions which exist only for negative time.

APPENDIX C

A VARIATIONAL METHOD FOR DERIVING W-H EQUATIONS

Suppose the integrand of the performance index (or a particular element of the index) has the form

$$\Phi = X_* H_* H Y \tag{C-1}$$

the first variation is found by taking the gradient with respect to H_*:

$$\nabla\Phi_{H_*} = X_*'[HY]' = X'(-s)Y'(-s)H'(s) \tag{C-2}$$

<u>Example 1:</u>

From Section II, the regulator problem gave

$$\Phi = X_* Q X + U_* R U \tag{C-3}$$

with

$$X = AU + Bx(0) \tag{C-4}$$

Therefore,

$$\Phi = [U_* A_* + x'(0)B_*]\, Q\, [AU + Bx(0)] + U_* R U \tag{C-5}$$

Taking the gradient with respect to U_* gives:

$$\nabla\Phi_{U_*} = [A_*]'[Q(AU + Bx(0)]' + (RU)' \tag{C-6}$$

Take the transpose and write

$$[R + A_* Q A]\, U_o + A_* Q B x(0) = \psi \tag{C-7}$$

verifying Eq. 6.

Example 2:

From Section III (see Eq. 41), since [setting $x(0) = 0$]

$$E = \hat{X} - X = W_a[V + HBN] - BN \qquad (C-8)$$

we write

$$\Phi = [(N_*B_*H_* + V_*)W_{a*} - N_*B_*][E] \qquad (C-9)$$

Take the gradient with respect to W_{a*}:

$$\nabla\Phi W_{a*} = [N_*B_*H_* + V_*]'[E]' \qquad (C-10)$$

Choosing to work with the transpose gives

$$[W_a(V + HBN) - BN][N_*B_*H_* + V_*] \qquad (C-11)$$

Expanding gives:

$$\nabla\Phi = W_a[VN_*B_*H_* + VV_* + HBNN_*B_*H_* + HBNV_*] - BNN_*B_*H_* - BNV_* \qquad (C-12)$$

The expectation operation on Eq. C-12 gives

$$W_a[\varphi_{\overline{VN}'}B_*H_* + \varphi_{\overline{VV}'} + HB\varphi_{\overline{NN}'}B_*H_* + HB\varphi_{\overline{NV}'}] - B\varphi_{\overline{NN}'}B_*H_* - B\varphi_{\overline{NV}'} = \psi \qquad (C-13)$$

If V and N are independent, Eq. (C-13) reduces to

$$W_a[\varphi_{\overline{VV}'} + HB\varphi_{\overline{NN}'}B_*H_*] - B\varphi_{\overline{NN}'}B_*H_* = \psi \tag{C-14}$$

This verifies Eq. 42.

Example 3:

Equation 67,

$$\Phi = X_*QX + U_*RU \tag{C-15}$$

with

$$U = W_a[V + HBN] \tag{C-16}$$

$$X = AW_a[V + HBN] + BN \tag{C-17}$$

becomes

$$\begin{aligned}\Phi = {} & [N_*B_* + (N_*B_*H_* + V_*)W_{a*}A_*]Q[AW_a(V + HBN) + BN] \\ & + [N_*B_*H_* + V_*]\, W_{a*}R\, W_a\, [V + HBN]\end{aligned} \tag{C-18}$$

Therefore

$$\begin{aligned}\nabla\varphi_{W_{a*}} = {} & [N_*B_*H_* + V_*]' \Big\{ [A_*Q(AW_a(V + HBN) + BN)]' \\ & + [RW_a(V + HBN)]' \Big\}\end{aligned} \tag{C-19}$$

Again, we prefer to work with the transpose

$$(\nabla\varphi_{W_{a*}})' = \Big\{A_*QAW_a[V + HBN] + A_*QBN + RW_a[V + HBN]\Big\}[N_*B_*H_* + V_*] \quad \text{(C-20)}$$

$$= [R + A_*QA]W_a[V + HBN][N_*B_*H_* + V_*] + A_*QBN[N_*B_*H_* + V_*] \quad \text{(C-21)}$$

Given $\varphi_{\overline{VN}}' = \varphi_{\overline{NV}}' = 0$, then taking the expectation yields

$$[R + A_*QA]W_a[\varphi_{\overline{VV}}' + HB\varphi_{\overline{NN}}'B_*H_*] + A_*QB\varphi_{\overline{NN}}'B_*H_* = \psi \quad \text{(C-22)}$$

verifying Eq. 71.

APPENDIX D

THE OUTPUT REGULATOR

The system is linear and time invariant:

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned} \tag{D-1}$$

Transforming Eq. D-1 gives

$$\begin{aligned} X &= [Is - F]^{-1} GU + [Is - F]^{-1} x(0) \\ Y &= HX \end{aligned} \tag{D-2}$$

The block diagram, assuming a control law,

$$y = -KX \quad \text{(K may be frequency dependent)}$$

is shown in Fig. D-1.

From Fig. D-1, write

$$U = -[I + KHA]^{-1} KHB\,x(0) = W_a HB\,x(0) \tag{D-3}$$

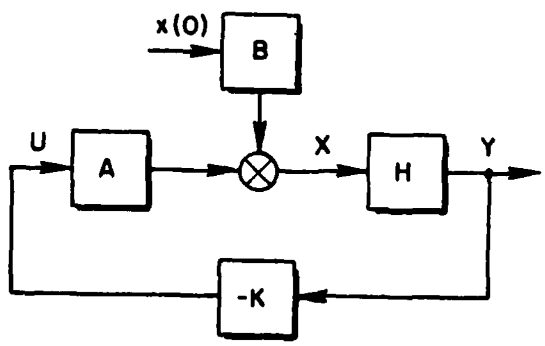


Figure D-1. Closed-Loop Configuration

Minimizing the quadratic index,

$$J = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} (X_* QX + U_* RU)\, ds \quad , \quad X_* = X'(-s) \tag{D-4}$$

with respect to W_a gives the W-H equation as:

$$[R + A_* QA] W_a (HB\, x(0)\, x'(0)\, B_* H_*) + A\ QB\, x(0)\, x'(0)\, B_* H_* = \psi \tag{D-5}$$

Equation D-5 can be solved using the direct approach or spectral factorization. Using the factorization approach, write Eq. D-5 as

$$[F_* RF] W_a\, G\, G_* + N = \psi \tag{D-6}$$

so that

$$W_a = -F^{-1} R^{-1} [F_*^{-1} N G_*^{-1}]_+ G^{-1} \tag{D-7}$$

F can be computed using the "full state" feedback regulator gains:

$$F = I + KA \tag{D-8}$$

for example:

$$x = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 5 \end{bmatrix} U \quad , \quad H = [\,1 \quad 0\,] \tag{D-9}$$

Therefore,

$$X(s) = \frac{\begin{bmatrix} s+1 \\ 5s+1 \end{bmatrix}}{s(s-3)} U + \frac{\begin{bmatrix} s-4 & 1 \\ -4 & s+1 \end{bmatrix}}{s(s-3)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{D-10}$$

Let

$$R = 1 \quad , \quad Q = \begin{bmatrix} \frac{28}{75} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \tag{D-11}$$

so that

$$\det[R + A_* QA] \;\propto\; s^2 + 1.2s + .2 = (s + .2)(s + 1) \tag{D-12}$$

Also note

$$\det[HB\,\mathbf{x}(0)\,\mathbf{x}'(0)\,B_* H_*] = \frac{x_1^2(0)\left[-s^2 + \left(\frac{x_2(0) - 4x_1(0)}{x_1(0)}\right)\right]^2}{D\,\overline{D}}$$

$$= \frac{x_1^2(0)(-s + \alpha)(s + \alpha)}{D\,\overline{D}} \tag{D-13}$$

Letting

$$\alpha = \left| \frac{x_2(0) - 4x_1(0)}{x_1(0)} \right| \tag{D-14}$$

gives

$$W_a = \frac{-(\alpha + 3.2)\left[s + \frac{.2\alpha}{\alpha + 3.2}\right] D}{(s + \alpha)\Delta} \tag{D-15}$$

$$K = \frac{(\alpha + 3.2)\left[s + \frac{.2\alpha}{\alpha + 3.2}\right]}{s + 1} \tag{D-16}$$

$$U = \frac{-(\alpha+3.2)\left[s + \frac{.2\alpha}{\alpha+3.2}\right]\left[\, s-4 \mid 1 \,\right]}{(s+\alpha)\Delta}\, x(0) \tag{D-17}$$

$$X = \begin{bmatrix} \frac{s-4}{(s+.2)(s+\alpha)} & \frac{1}{(s+.2)(s+\alpha)} \\ \\ \frac{-(5\alpha+20)}{(s+1)(s+\alpha)} & \frac{s+(5+\alpha)}{(s+1)(s+\alpha)} \end{bmatrix} \begin{bmatrix} x_1(0) \\ \\ x_2(0) \end{bmatrix} \tag{D-18}$$

Note that $x_1(0) \equiv 0$ gives $K \equiv \infty$.

Given perfect knowledge of $x_1(0)$, $x_2(0)$, observe

$$X = \begin{bmatrix} \frac{(s-4)x(0) + x_2(0)}{(s+.2)(s+\alpha)} \\ \hline \frac{-(5\alpha+20)x(0) + [s+(s+\alpha)]x(0)}{(s+1)(s+\alpha)} \end{bmatrix} \equiv \begin{bmatrix} \frac{x_1(0)}{s+.2} \\ \\ \frac{x_2(0)}{s+.10} \end{bmatrix} \tag{D-19}$$

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